

Matrix models for 5d super Yang-Mills

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Abstract

In this contribution to the review on localization in gauge theories we investigate the matrix models derived from localizing $\mathcal{N} = 1$ super Yang-Mills on S^5 . We consider the large- N limit and attempt to solve the matrix model by a saddle-point approximation. In general it is not possible to find an analytic solution, but at the weak and the strong limits of the 't Hooft coupling there are dramatic simplifications that allows us to extract most of the interesting information. At weak coupling we show that the matrix model is close to the Gaussian matrix model and that the free-energy scales as N^2 . At strong coupling we show that if the theory contains one adjoint hypermultiplet then the free-energy scales as N^3 . We also find the expectation value of a supersymmetric Wilson loop that wraps the equator. We demonstrate how to extract the effective couplings and reproduce results of Seiberg. Finally, we compare to results for the six-dimensional $(2, 0)$ theory derived using the AdS/CFT correspondence. We show that by choosing the hypermultiplet mass such that the supersymmetry is enhanced to $\mathcal{N} = 2$, the Wilson loop result matches the analogous calculation using AdS/CFT. The free-energies differ by a rational fraction.

This is a contribution to the review volume “Localization techniques in quantum field theories” (eds. V. Pestun and M. Zabzine) which contains 17 Chapters available at [1]

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1 Introduction

In this installment of the review on localization we analyze the matrix models that result from localizing five-dimensional $\mathcal{N} = 1$ super Yang-Mills on a five-sphere of radius r . In five dimensions

the supermultiplets have one vector multiplet and some hypermultiplets. In this generic case there are a total of eight supersymmetries. The most interesting case for us is when there is one hypermultiplet in the adjoint representation with a particular mass, M . We will refer to such theories as $\mathcal{N} = 1^*$ theories. When $M = i/(2r)$ the supersymmetry is enhanced to $\mathcal{N} = 2$, with 16 total supersymmetries. This is the maximal amount of supersymmetry in five dimensions (without gravity), so we will refer to this as maximally supersymmetric Yang-Mills (MSYM).

The reason that the $\mathcal{N} = 2$ enhancement is interesting is that the mysterious six-dimensional $(2, 0)$ superconformal field theory when compactified on a circle reduces to MSYM, with the radius related to the Yang-Mills coupling by

$$R_6 = \frac{g_{YM}^2}{8\pi^2} . \quad (1.1)$$

This relation follows from identifying the Kaluza-Klein modes of the $(2, 0)$ theory with the instanton particles in the 5d MSYM [2]. These $(2, 0)$ theories are difficult to study because they have no free parameters and no Lagrangian description, and thus no perturbative prescription. However, they do have an $AdS_7 \times S^4$ dual, so certain strong coupling data can be extracted using supergravity. The hope then is that one can use the localization results from the MSYM to say something about the $(2, 0)$ theory. For example, one can now say much about the supersymmetric indices of the $(2, 0)$ theories using MSYM (see Contribution [3]).

One thing to keep in mind about this discussion is that the MSYM is non-renormalizable and hence requires a UV completion. The $(2, 0)$ theory on the circle is believed to be a consistent UV completion¹. The observables we compute using localization are however finite because of the supersymmetry and would be expected to match with the same observables in the UV complete theory.

Localization results in a complicated matrix model that is not analytically solvable in general. However, we will show that in the large- N limit at strong coupling the analysis of the matrix model simplifies dramatically. One of the main results is that free-energy scales as N^3 [9, 10] with a coefficient that depends on M [10]. The supergravity analysis of the $(2, 0)$ theory also exhibits N^3 behavior for the free-energy [11, 12], suggesting that the degrees of freedom are more than for a weakly coupled gauge theory, where one finds N^2 scaling in the free-energy. However, at the MSYM point, the coefficient in front of the N^3 term differs by a factor of 4/5 with the N^3 term in the supergravity calculation. This remains an unresolved problem.

Nevertheless one can study another supersymmetric observable, the expectation value of a Wilson loop along the equator. Here one finds a match at the MSYM point with a parallel computation done using supergravity [13].

The rest of the paper is organized as follows. In section 2 we give some details of the matrix model resulting from localization of 5d SYM and study limits at large volume or large hypermultiplet mass, reproducing results in [14, 15] for the effective couplings. In section 3 we consider the large N behavior of the $\mathcal{N} = 1^*$ theories. We calculate the free energy and the expectation value of a supersymmetric Wilson loop in the weak and strong coupling limits. We also generalize these results to \mathbb{Z}_k quiver theories. In section 4 we compute the free energy and the Wilson loop expectation value starting from the supergravity dual of the $(2, 0)$ theory on $S^5 \times R$, and then compactifying the R to an S^1 and identifying the radius as in (1.1). We show that there is a mismatch with the

¹It had been proposed that MSYM could be used to actually define the $(2, 0)$ theories [4, 5, 6], and while not renormalizable, might actually be finite [4]. However, an explicit calculation shows that the four-point amplitude is UV divergent at six loops [7] and hence requires a UV completion. For a possible way around this see [8].

free-energy result from section 3 by a factor of 4/5, but the Wilson loop results agree. In section 5 we give a brief summary discuss some open problems.

2 Matrix model for $\mathcal{N} = 1$ 5d Yang-Mills with matter

The perturbative partition function was derived in [16] for massless hypermultiplets and in [17] for MSYM. Its derivation is given in Contribution [18]. In this section we show how the results of the effective couplings in [14, 15] can be extracted from the resulting matrix model.

We consider a theory with a semi-simple compact gauge group G . This has an $\mathcal{N} = 1$ vector multiplet and $\mathcal{N} = 1$ massless hypermultiplets in representation R_i with splittings into half-multiplets when R_i is complex. The partition function of this gauge theory on S^5 is then given by

$$Z = \int_{\text{Cartan}} [d\phi] e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) - \frac{\pi k}{3} \text{Tr}(\phi^3)} Z_{1\text{-loop}}^{\text{vect}}(\phi) \prod_i Z_{1\text{-loop}}^i(\phi) + \mathcal{O}(e^{-\frac{16\pi^3 r}{g_{YM}^2}}), \quad (2.1)$$

where the one-loop contributions are given by the infinite products

$$Z_{1\text{-loop}}^{\text{vect}}(\phi) = \prod_{\beta} \prod_{t \neq 0} (t - \langle \beta, i\phi \rangle)^{(1 + \frac{3}{2}t + \frac{1}{2}t^2)}, \quad (2.2)$$

and

$$Z_{1\text{-loop}}^i(\phi) = \prod_{\mu \in R_i} \prod_t \left(t - \langle i\phi, \mu \rangle + \frac{3}{2} \right)^{-(1 + \frac{3}{2}t + \frac{1}{2}t^2)}, \quad (2.3)$$

with β the roots and μ the weights in R_i .

The path integral in (2.1) has a contribution from a Chern-Simons term with level k . We have also absorbed the radius r into the integration variable $\phi = -ir\sigma$. As in the 4D case [19], we must Wick rotate and integrate over real ϕ in order to have a well-defined path integral.

The infinite products that appear in (2.2) and (2.3) are divergent and need to be regularized. Each one-loop contribution has the form

$$\mathcal{P} = x \prod_{t=1}^{\infty} (t+x)^{(1 + \frac{3}{2}t + \frac{1}{2}t^2)} (t-x)^{(1 - \frac{3}{2}t + \frac{1}{2}t^2)}, \quad (2.4)$$

whose log can be written as

$$\log \mathcal{P} = \sum_{t=1}^{\infty} \left(3x - \frac{x^2}{2} \right) + \text{convergent part}. \quad (2.5)$$

Therefore, the infinite product can be regulated by replacing it with the triple sine function [20]

$$S_3(x) = 2\pi e^{-\zeta'(-2)} x e^{\frac{x^2}{4} - \frac{3}{2}x} \prod_{t=1}^{\infty} \left(\left(1 + \frac{x}{t} \right)^{(1 + \frac{3}{2}t + \frac{1}{2}t^2)} \left(1 - \frac{x}{t} \right)^{(1 - \frac{3}{2}t + \frac{1}{2}t^2)} e^{\frac{x^2}{2} - 3x} \right), \quad (2.6)$$

As an alternative we can regularize the divergence by introducing a UV cut-off that stops the mode expansion at $n_0 = \pi\lambda r$, and leaving the log of the one-loop determinants to be

$$\begin{aligned} \log(Z_{1\text{-loop}}^{\text{vect}}(\phi) \prod_i Z_{1\text{-loop}}^i(\phi)) &= -\frac{\pi\lambda r}{2} \sum_{\beta} (\langle\beta, i\phi\rangle)^2 + \frac{\pi\lambda r}{2} \sum_{\mu \in R_i} (\langle i\phi, \mu\rangle)^2 + \text{convergent part} \\ &= \pi\lambda r \left(C_2(\text{adj}) - \sum_i C_2(R_i) \right) \text{Tr}(\phi^2) + \text{convergent part}, \end{aligned} \quad (2.7)$$

where $\text{Tr}(T_A T_B) = C_2(R) \delta_{AB}$ and $\sum_{\mu \in R_i} (\langle\phi, \mu\rangle)^2 = 2C_2(R_i) \text{Tr}(\phi^2)$. The linear piece cancels since the gauge group is semi-simple. Hence, the divergent piece is proportional to $\text{Tr}(\phi^2)$ and can be absorbed into an effective coupling given by

$$\frac{1}{g_{eff}^2} = \frac{1}{g_{YM}^2} - \frac{\lambda}{8\pi^2} \left(C_2(\text{adj}) - \sum_i C_2(R_i) \right). \quad (2.8)$$

This renormalization of the coupling agrees with the flat space results in [21, 22].

The convergent part of (2.4) can be replaced by $S_3(x)e^{-\frac{x^2}{4} + \frac{3}{2}x}$ up to x -independent (and hence irrelevant) constants. The extra exponential factor leads to a further finite shift in the coupling constant. Notice that the UV divergence cancels if there is only one hypermultiplet and it sits in the adjoint representation.

Using the regularized determinants, we can rewrite the matrix model in terms of triple sine functions $S_3(x)$

$$Z = \int d\phi e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) - \frac{\pi k}{3} \text{Tr}(\phi^3)} \det_{Ad} \left(S_3(i\phi) \right) \prod_i \det_{R_i}^{-1} \left(S_3 \left(i\phi + \frac{3}{2} \right) \right), \quad (2.9)$$

where from now on we assume that g_{YM} is the renormalized coupling. The triple sine function $S_3(x)$ has the symmetry property

$$S_3(-x) = S_3(x+3), \quad S_3 \left(x + \frac{3}{2} \right) = S_3 \left(-x + \frac{3}{2} \right). \quad (2.10)$$

The weights are mapped from μ to $-\mu$ when exchanging representation R with \bar{R} . Hence, the one-loop contribution of a massless hypermultiplet has the property

$$\det_R \left(S_3 \left(i\phi + \frac{3}{2} \right) \right) = \det_R \left(S_3 \left(-i\phi + \frac{3}{2} \right) \right) = \det_{\bar{R}} \left(S_3 \left(i\phi + \frac{3}{2} \right) \right). \quad (2.11)$$

and the representations R and \bar{R} are automatically symmetrized in the determinants.

Hypermultiplet masses can be turned on by using an auxiliary $U(1)$ vector multiplet. One takes a $G \times U(1)$ matrix model, but excludes the integration over the $U(1)$ direction. Thus the contribution of massive hypermultiplets is given by

$$Z = \int d\phi e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) - \frac{\pi k}{3} \text{Tr}(\phi^3)} \det_{Ad} \left(S_3(i\phi) \right) \prod_i \det_{R_i}^{-1} \left(S_3 \left(i\phi + im_i + \frac{3}{2} \right) \right), \quad (2.12)$$

where m_i are dimensionless parameters related to the actual hypermultiplet masses by $m_i = rM_i$. Using the triple sine's symmetry we find the relation

$$\det_{\mathbf{R}_i} \left(S_3 \left(i\phi + im_i + \frac{3}{2} \right) \right) = \det_{\bar{\mathbf{R}}_i} \left(S_3 \left(i\phi - im_i + \frac{3}{2} \right) \right). \quad (2.13)$$

Hence, the partition function with massive hypermultiplets can be written as

$$\int d\phi e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) - \frac{\pi k}{3} \text{Tr}(\phi^3)} \det_{Ad} \left(S_3(i\phi) \right) \times \prod_i \det_{\mathbf{R}_i}^{-1/2} \left(S_3 \left(i\phi + im_i + \frac{3}{2} \right) \right) \det_{\bar{\mathbf{R}}_i}^{-1/2} \left(S_3 \left(i\phi - im_i + \frac{3}{2} \right) \right). \quad (2.14)$$

Let us consider (2.14) in the large volume limit by taking $r \rightarrow \infty$. We can write (2.14) in the form

$$\int d\phi e^{-\mathcal{F}}, \quad (2.15)$$

where

$$\mathcal{F} = \frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) + \frac{\pi k}{3} \text{Tr}(\phi^3) - \sum_{\beta} \log S_3(\langle i\phi, \beta \rangle) + \sum_i \sum_{\mu_i} \log S_3 \left(\langle i\phi, \mu_i \rangle + im_i + \frac{3}{2} \right).$$

We then restore the r dependence by the rescaling $\phi \rightarrow r\phi$ and $m \rightarrow rm$. Using the asymptotic expansion for $|\text{Im}z| \rightarrow \infty$ and $0 \leq \text{Re}z < 3$

$$\log S_3(z) \sim -\text{sgn}(\text{Im}z) \pi i \left(\frac{1}{6} z^3 - \frac{3}{4} z^2 + z + \dots \right), \quad (2.16)$$

we obtain the expression

$$\frac{1}{2\pi r^3} \mathcal{F} = \frac{4\pi^2}{g_{YM}^2} \text{Tr}(\phi^2) + \frac{k}{6} \text{Tr}(\phi^3) + \frac{1}{12} \left(\sum_{\beta} |\langle \phi, \beta \rangle|^3 - \sum_i \sum_{\mu_i} |\langle \phi, \mu_i \rangle + m_i|^3 \right) + O(r^{-2}). \quad (2.17)$$

Up to a constant which we have absorbed into the definition of the coupling, (2.17) matches the quantum prepotential in the flat-space limit [15]. The normalization of the quadratic term is fixed either by a direct one-loop calculation in flat space [22] or by matching the superpotential in 5d with the corresponding one in 4D [21].

The matrix model is well-defined if \mathcal{F} is a convex positive function in the large ϕ limit. In this limit \mathcal{F} takes the asymptotic form

$$\begin{aligned} \mathcal{F} = & \frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) + \frac{\pi k}{3} \text{Tr}(\phi^3) + \frac{\pi}{6} \left(\sum_{\beta} |\langle \phi, \beta \rangle|^3 - \sum_i \sum_{\mu_i} |\langle \phi, \mu_i \rangle|^3 \right) \\ & - \pi \sum_{\beta} |\langle \phi, \beta \rangle| - \frac{\pi}{2} \sum_i \sum_{\mu_i} \left(m_i \text{sgn}(\langle \phi, \mu_i \rangle) (\langle \phi, \mu_i \rangle)^2 + (m_i^2 + \frac{1}{4}) |\langle \phi, \mu_i \rangle| \right) + \dots, \end{aligned} \quad (2.18)$$

where the ellipsis stands for terms suppressed at large ϕ . Analyzing the convexity of (2.18) it is clear that the cubic terms dominate. Hence, the analysis is identical to that in [15] and the same

conditions apply. In special cases the cubic terms cancel each other, for example in the case of single adjoint hypermultiplet [10] or for the superconformal $USp(2N)$ theory described in [23].

Suppose we now take the hypermultiplet masses to infinity. For large $|m_i|$ the leading terms in (2.16) are

$$\begin{aligned} \mathcal{F} = & \frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2) + \frac{\pi k}{3} \text{Tr}(\phi^3) - \sum_{\beta} \log S_3(\langle i\phi, \beta \rangle) \\ & - \sum_i \text{sgn}(m_i) \frac{\pi}{2} \sum_{\mu_i} \left(\frac{1}{3} (\langle \phi, \mu_i \rangle)^3 + m (\langle \phi, \mu_i \rangle)^2 \right). \end{aligned}$$

The two last terms in (2.19) can be absorbed by a redefinition of k and g_{YM} . To see this, note that

$$\text{Tr}(T_A T_B T_C + T_A T_C T_B) = C_3(\mathbf{R}) d_{ABC}, \quad (2.19)$$

where

$$\sum_{\mu} (\langle \phi, \mu \rangle)^3 = C_3(\mathbf{R}) \text{Tr}(\phi^3). \quad (2.20)$$

The coefficient C_3 satisfies $C_3(\bar{\mathbf{R}}) = -C_3(\mathbf{R})$, hence it is zero for real representations. For the lower complex representations in $SU(N)$ it is 1 for the fundamental, $N - 4$ for the antisymmetric, and $N + 4$ for the symmetric representations. Hence, from (2.19) and (2.20) we get

$$k_{eff} = k - \sum_i \text{sgn}(m_i) \frac{C_3(\mathbf{R}_i)}{2}, \quad (2.21)$$

the same result in [14, 15]. A similar analysis of the quadratic terms gives

$$\frac{r}{g_{eff}^2} = \frac{r}{g_{YM}^2} - \sum_i \frac{|m_i|}{8\pi^2} C_2(\mathbf{R}_i). \quad (2.22)$$

3 $\mathcal{N} = 1^* 5d$ super Yang-Mills

We now turn to $\mathcal{N} = 1^*$ super Yang-Mills, where there is a single adjoint hypermultiplet with mass parameter m . We further assume that the gauge group is $SU(N)$.

To analyze the resulting matrix model (2.14) we rewrite the triple sine function $S_3(z)$ in (2.6) as

$$S_3(z) = 2e^{-\zeta'(-2)} \sin(\pi z) e^{\frac{1}{2}f(z)} e^{\frac{3}{2}l(z)}, \quad (3.1)$$

where $l(z)$ and $f(z)$ are given by [24, 25]

$$l(z) = -z \log(1 - e^{2\pi iz}) + \frac{i}{2} \left(\pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz}) \right) - \frac{i\pi}{12} \quad (3.2)$$

$$f(z) = \frac{i\pi z^3}{3} + z^2 \log(1 - e^{-2\pi iz}) + \frac{iz}{\pi} \text{Li}_2(e^{-2\pi iz}) + \frac{1}{2\pi^2} \text{Li}_3(e^{-2\pi iz}) - \frac{\zeta(3)}{2\pi^2}. \quad (3.3)$$

While these functions are rather ugly, their derivatives have the much nicer form

$$\frac{df(z)}{dz} = \pi z^2 \cot(\pi z); \quad \frac{dl(z)}{dz} = -\pi z \cot(\pi z); \quad (3.4)$$

The matrix model path integral (2.14) can then be rewritten as

$$Z = \int_{Cartan} [d\phi] e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}(\phi^2)} \prod_{\beta} (\sin(\pi \langle \beta, i\phi \rangle) e^{-\frac{1}{4}l(\frac{1}{2}-im-\langle \beta, i\phi \rangle) - \frac{1}{4}l(\frac{1}{2}-im+\langle \beta, i\phi \rangle)} \\ \times e^{\frac{1}{2}f(\langle \beta, i\phi \rangle) - \frac{1}{4}f(\frac{1}{2}-im-\langle \beta, i\phi \rangle) - \frac{1}{4}f(\frac{1}{2}-im+\langle \beta, i\phi \rangle)} + \dots), \quad (3.5)$$

up to instanton terms, where we have dropped the Chern-Simons term. From now on we assume that the gauge group is $U(N)$. Defining the t' Hooft coupling constant to be

$$\lambda = \frac{g_{YM}^2 N}{r},$$

and taking the large N limit for fixed λ , all instanton contributions are suppressed. We can then re-express (3.5) as the integral over the eigenvalues ϕ_i

$$Z \sim \int \prod_{i=1}^N d\phi_i \exp \left(-\frac{8\pi^3 N}{\lambda} \sum_i \phi_i^2 + \sum_{j \neq i} \sum_i \left[\log [\sinh(\pi(\phi_i - \phi_j))] \right. \right. \\ \left. \left. - \frac{1}{4}l \left(\frac{1}{2} - im + i(\phi_i - \phi_j) \right) - \frac{1}{4}l \left(\frac{1}{2} - im - i(\phi_i - \phi_j) \right) + \frac{1}{2}f(i(\phi_i - \phi_j)) - \right. \right. \\ \left. \left. - \frac{1}{4}f \left(\frac{1}{2} - im + i(\phi_i - \phi_j) \right) - \frac{1}{4}f \left(\frac{1}{2} - im - i(\phi_i - \phi_j) \right) \right] \right). \quad (3.6)$$

In the large N limit the partition function in (3.6) is dominated by the saddle point. Using the derivatives in (3.4), the ϕ_i at the saddle point satisfy

$$\frac{16\pi^3 N}{\lambda} \phi_i = \pi \sum_{j \neq i} \left[(2 - (\phi_i - \phi_j)^2) \coth(\pi(\phi_i - \phi_j)) \right. \\ \left. + \frac{1}{2} \left(\frac{1}{4} + (\phi_i - \phi_j - m)^2 \right) \tanh(\pi(\phi_i - \phi_j - m)) \right. \\ \left. + \frac{1}{2} \left(\frac{1}{4} + (\phi_i - \phi_j + m)^2 \right) \tanh(\pi(\phi_i - \phi_j + m)) \right]. \quad (3.7)$$

In general this equation is not solvable, but it simplifies a lot both at weak ($\lambda \ll 1$) and strong ($\lambda \gg 1$) coupling.

For weak coupling, the contribution from the hypermultiplet plays no role and (3.7) reduces to

$$\frac{16\pi^3 N}{\lambda} \phi_i \approx 2 \sum_{j \neq i} \frac{1}{\phi_i - \phi_j}. \quad (3.8)$$

This is the same equation one finds for a Gaussian matrix model and in the large- N limit its solution has the Wigner distribution

$$\rho(\phi) \equiv \frac{1}{N} \frac{dn}{d\phi} = \frac{2}{\pi \phi_0^2} \sqrt{\phi_0^2 - \phi^2}, \quad \phi_0 = \sqrt{\frac{\lambda}{4\pi^3}}, \quad (3.9)$$

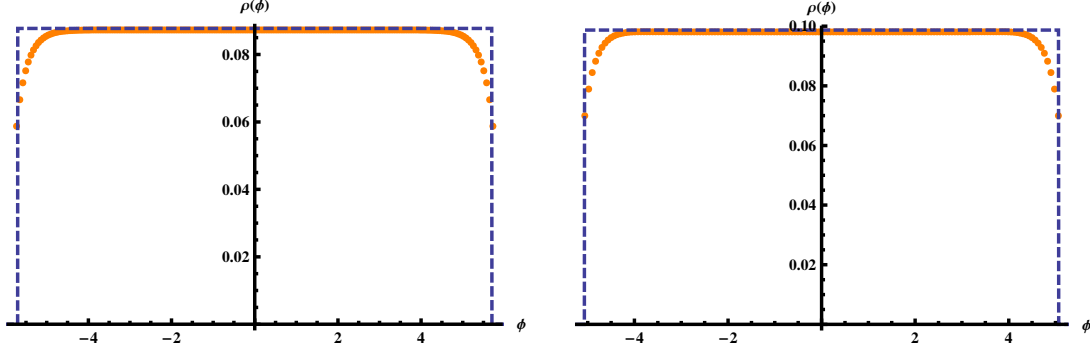


Figure 1: Numerical results for the eigenvalue density $\rho(\phi)$ for (left) $m = 0$, $N = 200$, $\lambda = 400$ and (right) $m = \frac{1}{2}$, $N = 160$, $\lambda = 320$. The dashed blue lines are the strong coupling solutions in (3.14).

where the eigenvalue density is normalized to

$$\int \rho(\phi) d\phi = 1. \quad (3.10)$$

The free energy then has the typical weak coupling form

$$F = -\log Z \approx -N^2 \log \sqrt{\lambda}. \quad (3.11)$$

At strong coupling and with $|m| \ll \lambda$ we can simplify (3.7) by making the ansatz $|\phi_i - \phi_j| \gg 1$. In general this is not the case for every pair (i, j) , but it will be true for most of them. The saddle point equation then simplifies to

$$\frac{16\pi^3 N}{\lambda} \phi_i = \pi \left(\frac{9}{4} + m^2 \right) \sum_{j \neq i} \text{sgn}(\phi_i - \phi_j). \quad (3.12)$$

If we assume the ϕ_i are ordered, we get

$$\phi_i = \frac{(9 + 4m^2) \lambda}{64\pi^2 N} (2i - N). \quad (3.13)$$

Hence the eigenvalue density is constant over its support,

$$\begin{aligned} \rho(\phi) &= \frac{32\pi^2}{(9 + 4m^2) \lambda}, & |\phi| \leq \phi_m, & \quad \phi_m = \frac{(9 + 4m^2) \lambda}{64\pi^2} \\ &= 0 & |\phi| > \phi_m. \end{aligned} \quad (3.14)$$

In figure 1 we compare the approximation in (3.14) with numerical results using the full kernel in (3.7). As one can see the approximation is very good except with a slight deterioration at the end-points.

Using the strong-coupling ansatz, the partition function in (3.5) simplifies to

$$Z \sim \int \prod_i d\phi_i e^{-\frac{8\pi^3 N}{\lambda} \sum_i \phi_i^2 + \frac{\pi}{2} \left(\frac{9}{4} + m^2 \right) \sum_{j \neq i} \sum_i |\phi_i - \phi_j|}. \quad (3.15)$$

Applying the saddle point solution (3.13), we find for the free energy

$$F \equiv -\log Z \approx -\frac{g_{YM}^2 N^3}{96\pi r} \left(\frac{9}{4} + m^2 \right)^2 + \mathcal{O}(N^2). \quad (3.16)$$

Thus we see that the free energy crosses over from the N^2 dependence expected in a weakly coupled gauge theory to an N^3 behavior when going from weak to strong coupling.

The other interesting observable we can compute using localization is a supersymmetric Wilson loop [26, 27, 13]. Such loops in five-dimensional flat space were first considered in [28]. On S^5 , the loop must run along an S^1 fiber over a base $\mathbb{C}P^2$ in order to preserve some of the supersymmetries. The supersymmetric Wilson loop is given by

$$\langle W \rangle = \frac{1}{N} \left\langle \text{Tr} \left(\text{P exp} \left(i \oint_{S^1} (A_M v^M) \right) ds \right) \right\rangle. \quad (3.17)$$

where we have written the bosonic fields A_M in the 10D notation of [19, 29]. The vector v^M is the 10-dimensional vector $v^M = \epsilon \Gamma^M \epsilon$ where the components $M = 1, \dots, 5$ reduce to the Reeb vector on S^5 and $v^0 = 1$, where $A_0 = \sigma$. All other components of v^M are zero. Hence, along the localization locus in the zero instanton sector we have $A_M v^M = \sigma$. After Wick rotation the Wilson loop then becomes

$$\langle W \rangle = \frac{1}{N} \langle \text{Tr} e^{2\pi\phi} \rangle = \frac{1}{N} \left\langle \sum_i e^{2\pi\phi_i} \right\rangle. \quad (3.18)$$

In the large N limit the $\sum_i e^{2\pi\phi_i}$ term has a negligible back-reaction on the saddle point solutions. Thus in this limit the Wilson loop is well-approximated by the integral

$$\langle W \rangle = \int d\phi \rho(\phi) e^{2\pi\phi}. \quad (3.19)$$

At weak coupling where the density has support only for $|\phi| \ll 1$ we can approximate the integral as

$$\langle W \rangle \approx \int d\phi \rho(\phi) (1 + 2\pi^2 \phi^2) = 1 + \frac{\lambda}{8\pi} \approx \exp \left(\frac{\lambda}{8\pi} \right). \quad (3.20)$$

At strong coupling, where the density is approximately constant along its support, we find

$$\langle W \rangle \approx \frac{32\pi^2}{(9 + 4m^2)\lambda} \int_{-\phi_m}^{\phi_m} e^{2\pi\phi} d\phi \sim \exp \left(\frac{\lambda}{8\pi} \left(\frac{9}{4} + m^2 \right) \right), \quad (3.21)$$

where we have dropped the prefactor since it can be affected by our approximation and besides is not particularly important for the rest of the discussion. Interestingly, and unlike 4D, the argument in the exponent still has linear λ dependence at strong coupling, with the coefficient changed by the factor $(\frac{9}{4} + m^2)$ as compared to the weak coupling result.

The results for the free-energy and the Wilson loop can be generalized to a \mathbb{Z}_k quiver of the $\mathcal{N} = 1^*$ theory [10, 13]. The quiver has an $SU(N)^k$ gauge group with equal mass hypermultiplets

transforming in the bifundamental representations, $(N, \bar{N}, 1, \dots, 1)$, $(1, N, \bar{N}, 1, \dots)$, *etc.* The eigenvalues of (3.7) divide into k groups $\psi_i^{(r)}$, where $r = 1, \dots, k$, $i = 1, \dots, N$, resulting in the equations of motion

$$\begin{aligned} \frac{16\pi^3 N}{\lambda} \psi_i^{(r)} = & \pi \left[\sum_{j \neq i} \left(2 - (\psi_i^{(r)} - \psi_j^{(r)})^2 \right) \coth(\pi(\psi_i^{(r)} - \psi_j^{(r)})) \right. \\ & + \left(\sum_j \left[\frac{1}{4} \left(\frac{1}{4} + (\psi_i^{(r)} - \psi_j^{(r+1)} - m)^2 \right) \tanh(\pi(\psi_i^{(r)} - \psi_j^{(r+1)} - m)) \right. \right. \\ & \left. \left. + \frac{1}{4} \left(\frac{1}{4} + (\psi_i^{(r)} - \psi_j^{(r-1)} - m)^2 \right) \tanh(\pi(\psi_i^{(r)} - \psi_j^{(r-1)} - m)) \right] \right) \\ & \left. + (m \rightarrow -m) \right]. \end{aligned} \quad (3.22)$$

Equations (3.22) have a solution where $\psi_i^{(r)} = \psi_i^{(s)}$ for all r and s . Hence, if we take the same strong coupling ansatz as before, we find the eigenvalues set to $\psi_i^{(r)} = \phi_i$, where ϕ_i are the values in (3.13). The free energy is $F_k = kF$, where F is the free-energy in (3.16). The Wilson loop is the same as in (3.21).

4 Supergravity comparisons

In this section we compare the results for strongly coupled 5d SYM with the corresponding results found using the AdS/CFT correspondence [10, 13].

We begin by reviewing the supergravity computation of the free energy [10]. We consider supergravity on $AdS_7 \times S^4$ where the AdS_7 boundary is $S^1 \times S^5$. The radii of AdS_7 and S^4 are ℓ and $\ell/2$ respectively, where $\ell = 2\ell_{pl}(\pi N)^{1/3}$. The AdS_7 metric in global coordinates is given by

$$ds^2 = \ell^2 (\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_5^2), \quad (4.1)$$

where $d\Omega_5^2$ is the unit 5-sphere metric. The Euclidean time direction is compactified and has the identification $\tau \equiv \tau + 2\pi R_6/r$, while R_6 and r are the boundary radii of S^1 and S^5 .

Under the AdS/CFT correspondence, the supergravity classical action equals the free energy of the boundary field theory. The action needs to be regulated by adding counterterms [30, 31, 32, 33]. There can be scheme dependence in the regulation [32], but we will follow a minimal subtraction type prescription, which is the normal procedure when regulating the action. The full action then has the form

$$I_{AdS} = I_{\text{bulk}} + I_{\text{surface}} + I_{\text{ct}}, \quad (4.2)$$

where

$$I_{\text{bulk}} = -\frac{1}{16\pi G_N} \text{Vol}(S^4) \int d^7 x \sqrt{g} (R - 2\lambda) \quad (4.3)$$

is the action in the bulk with Newton's constant related to the 11-dimensional Planck length as $G_N = 16\pi^7 \ell_{pl}^9$. The other terms in (4.2) are the surface contribution I_{surf} and the counterterm I_{ct} , written in terms of the boundary metric and which cancels off divergences in I_{bulk} . One then finds the equations of motion

$$R - 2\lambda = -\frac{12}{\ell^2}, \quad (4.4)$$

and hence the action

$$I_{\text{bulk}} = -\frac{1}{256\pi^8 \ell_{pl}^9} \left(\frac{\pi^2 \ell^4}{6} \right) \frac{2\pi R_6}{r} \pi^3 (-12\ell^5) \int_0^{\rho_0} \cosh \rho \sinh^5 \rho d\rho = \frac{4\pi R_6}{3r} N^3 \sinh^6 \rho_0. \quad (4.5)$$

The integral diverges as $\rho_0 \rightarrow \infty$ and corresponds to a UV divergence in the boundary theory. We then set $\lambda = e^{\rho_0}$ where λ is the UV cutoff of the boundary theory, from which we obtain the expansion

$$\sinh^6 \rho_0 = \frac{1}{64} \lambda^6 - \frac{3}{32} \lambda^4 + \frac{15}{64} \lambda^2 - \frac{5}{16} + O(\lambda^{-2}). \quad (4.6)$$

The surface term contributes to the divergent pieces, but not the finite part of (4.5), while the counterterm cancels off the remaining divergent pieces with a minimal subtraction prescription. Hence, the regularized action is [31]

$$I_{AdS} = -\frac{5\pi R_6}{12r} N^3. \quad (4.7)$$

The Wilson loop expectation value can also be computed using supergravity [34]. Here, one considers an M2 brane that wraps the Euclidean time direction and the equator, while the brane's third direction drops straight down into the bulk. The Wilson loop is then dominated by the extremum of the world-volume action of the M2 brane,

$$\langle W \rangle \sim e^{-T^{(2)} \int dV}, \quad (4.8)$$

where the tension of the brane is $T^{(2)} = \frac{1}{(2\pi)^2 l_p^3}$. The M2 brane volume is given by

$$\int dV = l^3 \int_0^{\frac{2\pi R_6}{r}} d\tau \int_0^{2\pi} d\phi \int_0^{\rho_0} d\rho \sinh(\rho) \cosh(\rho) \quad (4.9)$$

Using the same UV cutoff as in (4.5) we find

$$T^{(2)} \int dV = \frac{\pi N R_6}{r} (\lambda - 2 + \lambda^{-1}). \quad (4.10)$$

The integral is again regulated using the minimal subtraction procedure and gives the regulated Wilson loop

$$\langle W \rangle \sim \exp \left(\frac{2\pi N R_6}{r} \right) \quad (4.11)$$

Using the identification in (1.1) we see that (4.11) matches with (3.21) if $m = i/2$, which is the enhancement point to $\mathcal{N} = 2$ supersymmetry [17, 29]. However, for this value of m the free-energy results in (4.7) and (3.16) differ by a factor of 5/4. Curiously, if we were to replace (1.1) with

$$R_6 = \frac{g_{YM}^2}{16\pi^2} \left(\frac{9}{4} + m^2 \right). \quad (4.12)$$

such that the Wilson loops agree for any value of m , then the free-energies in (4.7) and (3.16) agree for $m = 1/2$. At present we do not have an explanation for this.

We can also compare supergravity results for the quiver. This effectively replaces the S^4 with S^4/\mathbb{Z}_k , reducing the volume of this space by a factor of k , while at the same time replacing N with Nk . Therefore the free-energy becomes

$$I_{AdS} = -\frac{5\pi R_6 k^2}{12 r} N^3 . \quad (4.13)$$

Likewise, R_6 should be replaced with R_6/k . Hence we have the same sort of matching/mismatching between the gauge theory and supergravity as in the unquivered theory.

5 Discussion

In this review we have shown how to extract perturbative results for five-dimensional $\mathcal{N} = 1^*$ Yang-Mills theory on S^5 . There are of course many other interesting phenomena that one can explore. For example, one can squash the S^5 [35] which modifies the determinant factors in (2.3). Interestingly, the free-energy is the same as in (3.16), multiplied by the ratio of the squashed to unsquashed volume factors [35].

Also of interest are phase transitions. Here we will describe two types. The first is a transition between a Yang-Mills phase and a Chern-Simons phase [36]. Here it is possible to show that there is a third order phase transition when the ratio

$$\kappa \equiv \frac{8\pi^2}{k g_{YM}^2} \quad (5.1)$$

equals the critical value

$$\kappa_c = \sqrt{\frac{27\pi\lambda}{2}} \quad (5.2)$$

at weak coupling and

$$\kappa_c = 4\pi^2(9 + 4m^2)\lambda \quad (5.3)$$

at strong coupling.

Another type of phase transition occurs when taking the infinite volume limit [37], in a manner similar to what happens in four dimensions [38, 39], there occur an infinite number of transitions as the actual 't Hooft coupling, $t \equiv g_{YM}^2 N$, is taken to infinity. For an adjoint hypermultiplet with mass M , at strong coupling one finds a series of critical points at [37]

$$t_c^{(n)} = \frac{8\pi^2}{M}(n+1), \quad n \in \mathbb{Z}_+, \quad (5.4)$$

or in terms of the dimensionless 't Hooft coupling and mass parameter,

$$\lambda_c^{(n)} = \frac{8\pi^2}{m}(n+1), \quad n \in \mathbb{Z}_+. \quad (5.5)$$

An important open problem is the mismatch of the free-energies between the 5d gauge theory calculation and the supergravity computation of the (2,0) theory. One possibility is that this is a scheme dependence problem. In any event, we hope to see this discrepancy resolved in the near future.

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